S.-T. Yau College Student Mathematics Contests 2024

Oral Exams in Geometry and Topology

Team (Solve 2 out of 3 problems)

1.

(1) Let X and Y be compact, orientable manifolds of the same dimension. Define the degree of a continuous map $f : X \to Y$.

(2) Show that the degree of any continuos map $f : \mathbb{CP}^m \to \mathbb{CP}^m$ is of the form k^m for some integer k.

(3) Conversely, show that for any $k \in \mathbb{Z}_{>0}$, there is a continuous map $f : \mathbb{CP}^m \to \mathbb{CP}^m$ with degree k^m.

2. Show that the matrix multiplication $\vec{z} \cdot \vec{z}^*$, where $\vec{z}$ denotes a unit column vector in $\mathbb{C}^3$ and $\vec{z}^*$ is its conjugate transpose row vector, defines a $U(3)$-equivariant embedding from $\mathbb{CP}^2$ to the vector space of (3×3) Hermitian matrices.

Moreover, show that this is an isometric embedding. Here, $\mathbb{CP}^2$ is equipped with the standard Fubini-Study metric

$$\frac{1}{|\vec{z}|^4}\left(|\vec{z}|^2|d\vec{z}|^2 - (d\vec{z}^* \cdot \vec{z})(\vec{z}^* \cdot d\vec{z})\right).$$

The vector space of Hermitian matrices is equipped with the flat metric.

3. Let (S^2, g) be the two-dimensional unit sphere with the standard round metric g and ∇ be the covariant derivative associated with g. Suppose ω is a smooth differential 1-form on S^2 such that $\nabla\omega$ is proportional to g at each point.

(1) Find all such ω.

(2) What is the diffeomorphism (of S^2) generated by the dual of ω?